D-finite symmetric functions

Marni Mishna[1], Mariolys Rivas[1]

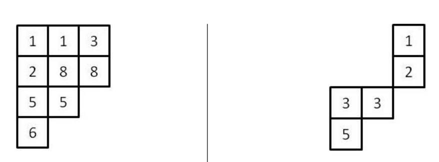
Department of Mathematics, Simon Fraser University

SFU

Partitions and Tableaux

- The *Young diagram* corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is an array of boxes such that the ith row has λ_i boxes.
- The *skew shape* λ/μ is the array of boxes given by the difference between the Young diagrams of λ and μ.
- A *semistandard Young tableau* (SSYT) is a Young diagram (or a skew shape) filled with positive integers which are weakly increasing in rows and strictly increasing in columns.

❀ **Example:** For $\lambda = (3,3,2,1)$ and $\mu = (2,2)$:



SSYT of shape λ. | Skew SSYT of shape λ/μ.

✿ **Problem:** For a given family of shapes of partitions, enumerate the SSYT. (eg. staircase, blocks, hooks).

Symmetric functions

Denote by Λ the vector space of symmetric functions over some field $\mathbb{K}$ over the basis of power sum symmetric functions:

$$p_\lambda(x_1, x_2, \ldots) := p_{\lambda_1} \cdots p_{\lambda_k},$$

such that $p_n = x_1^n + x_2^n + \cdots$.

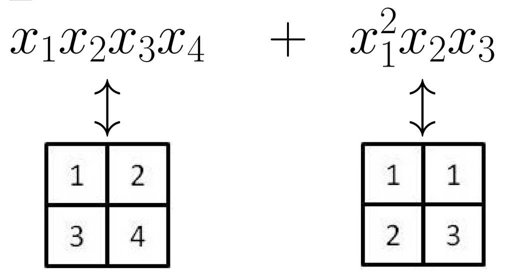
Other important bases are complete homogeneous (h_λ), elementary (e_λ) and Schur functions(s_λ). The Schur functions are known as the generating function for SSYT.

❀ **Example:**

$$e_3 = \sum_{i<j<k} x_i x_j x_k = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + \cdots .$$

$$h_3 = \sum_{i\leq j\leq k} x_i x_j x_k = x_1^2 + x_1x_2x_3 + x_1^2 x_2 + \cdots .$$

$$s_{(2,2)} = x_1x_2x_3x_4 \quad + \quad x_1^2 x_2 x_3 \quad + \quad \cdots$$



$\cdots$

$\cdots$

Operations on symmetric functions

- Scalar product $\langle\ ,\ \rangle$:
$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$
with $z_\lambda = 1^{r_1} 2^{r_2} \cdots k^{r_k} r_1! r_2! \cdots r_k!$, where $r_1, r_2, \ldots$ are the multiplicities of $1, 2, \ldots$ on λ.
- Kronecker product $*$:
$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$
- Adjoint multiplication: A homomorphism $D_f : \Lambda \to \Lambda$, such that for all $g_1, g_2 \in \Lambda$
$$\langle D_f g_1, g_2 \rangle = \langle g_1, f g_2 \rangle.$$

D-finiteness

$f \in \mathbb{K}[[x]]$ is *D-finite in* x.	The set $\{f', f'', \ldots\}$ spans a finite-dimensional vector subspace of $\mathbb{K}[[x]]$.
$f \in \mathbb{K}[[x_1, \ldots, x_n]]$ is *D-finite in* $\{x_1, \ldots, x_n\}$.	The set $\{\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1}, \ldots\}$ spans a finite-dimensional vector subspace of the space $\mathbb{K}[[x_1, \ldots, x_n]]$.
$f \in \mathbb{K}[[x_1, x_2, \ldots]]$ is *D-finite in* $\{x_1, x_2, \ldots\}$.	f is D-finite for any finite subset of $\{x_1, \ldots, x_n, \ldots\}$.
$f \in \Lambda$ is a *D-finite symmetric function.*	f is D-finite in $\{p_1, p_2, \ldots\}$.

Essentially D-finiteness corresponds to stating that f satisfies certain system of differential equations.

❀ **Examples:**
The function $f(t) = e^{t^2}$ is D-finite in t, since satisfies the differential equation $f' - 2tf = 0$.

The symmetric series $h(t) = \sum_n h_n t^n$ is D-finite since $h(t) = \exp\left(\sum_k \frac{p_k t^k}{k}\right)$.

Lemma: The symmetric series $h(t) = \sum_n h_n t^n$, $e(t) = \sum_n e_n t^n$ and $s(t) = \sum_{\lambda \in \mathcal{P}} s_\lambda t^{|\lambda|}$ are D-finite symmetric functions.

D-finiteness under the scalar product

The following important theorem of Gessel unites symmetric functions and enumeration:

Theorem [Gessel, 1990]

If f and g are D-finite symmetric functions (and D-finite in another variable t), **g involves only a finite number of p_i's**, and $\langle f, g \rangle$ is well-defined as a formal power series in t

$\Rightarrow \langle f, g \rangle$ is D-finite in t.

Corollary: If $\sum_{\lambda \in \mathcal{F}} s_{\lambda/\mu}$ is D-finite then the ordinary generating function $Y(t) = \langle \sum_{\lambda \in \mathcal{F}} s_{\lambda/\mu}, \sum_n h_j^n t^n \rangle$ for the number of SSYT of shape λ/μ for $\lambda \in \mathcal{F}$ and content of the form (j^n) is also D-finite.

We extended this to some new families of partitions by finding new D-finite symmetric functions, eg. $\mathcal{F} = \{(n^k, \lambda) : n \in \mathbb{N}\}$.

Extension of Gessel's theorem

The red condition in Gessel's theorem can be relaxed!

For example $\langle h, h(t) \rangle = \dfrac{1}{1-t}$ is D-finite in t.

We can bootstrap this result:

Theorem 1 [Mishna and Rivas, 2010]

Let f, g be D-finite series in the p_i's and another variable t, such that they both involve a finite number of p_i's. Then the scalar products

$\langle, \rangle$	$h(t)g$	$e(t)g$	$s(t)g$	$s(t)h(t)^{-1}g$	$s(t)e(t)^{-1}g$
hf	are D-finite functions				
ef	of t, whose DEs are computable.				

The proof uses adjoint multiplication and properties of D-finite functions.

Algebraic approach and D-finiteness

In the process of proving Theorem 1, we found a simpler proof of a Kronecker product formula.

Theorem 2 [Rivas, 2010]

$$(hf) * (hg) = h \sum_\gamma D_{p_\gamma}(f) D_{p_\gamma}(g) \frac{p_\gamma}{z_\gamma}.$$

The proof uses D-finiteness and we show that each side satisfies the same differential equation.

Open problems

❁ What are some sufficient conditions for the sum of all Schur functions indexed by the skew partitions in a given family $\mathcal{F}$ to be D-finite?
❁ What are necessary conditions for the scalar product $\langle f, g \rangle$ of two symmetric functions f and g to be D-finite?

References

[1] Frédéric Chyzak, Marni Mishna, and Bruno Salvy. Effective scalar products of D-finite symmetric functions. *Journal of Combinatorial Theory. Series A*, 112(1):1–43, 2005.

[2] Ira M. Gessel.Symmetric functions and P-recursiveness.*Journal of Combinatorial Theory. Series A*, 53(2):257–285, 1990.

[3] Jean-Yves Thibon. Hopf algebras of symmetric functions and tensor products of symmetric group representations. *International Journal of Algebra and Computation*, 1(2):207–221, 1991.

Funded in part by the NSERC and Graduate Fellowship.

1. Department of Mathematics, Simon Fraser University, 8888 University Road Burnaby BC Canada; mmishna@sfu.ca, mr42@sfu.ca.